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EQUATIONS OF STATE OF PIEZOCERAMIC SHELLS*

N.N. ROGACHEVA

The reduction of three-dimensional equations of electroelasticity to two-dimensional equations of piezoceramic shells is performed by an asymptotic method /1/ for the case of preliminary polarization along one of the families of coordinate lines of the middle surface. It is shown that for shells covered completely with electrodes on which a potential difference is maintained, the complete problem breaks down into a mechanical and an electrical problem, where the mechanical problem differs qualitatively from theories based on the Kirchhoff—Love type of hypotheses. For shells not covered with electrodes and loaded by a mechanical surface load, the complete problem does not generally allow of separation into mechanical and electrical probelems, in which connection the system of differential equations is of tenth order.

1. Let us select a system of tri-orthogonal coordinates $\alpha_1, \alpha_2, \gamma$ in such a manner that the lines α_1, α_2 would coincide with the lines of middle surface curvature, while the γ-lines would be orthogonal. In the coordinate system chosen we write the equations of electroelasticity of a piezoceramic shell which has first been polarized along the α_2-lines /2/:

The piezoeffect equations

$$\tau_i = n_{ii}\frac{a_j}{a_i}e_i + n_{ij}e_j - p_i\frac{\tau_3}{a_i} - c_i a_j E_2 \tag{1.1}$$

$$\tau_{ij} = \frac{1}{s_{44}^E}\left(\frac{a_i}{a_j}m_i + m_j\right) - \frac{d_{15}}{s_{44}^E}a_i E_1$$

$$\frac{\partial v_3}{\partial\gamma} = s_{12}^E\frac{\tau_1}{a_2} + s_{13}^E\frac{\tau_2}{a_1} + s_{11}^E\frac{\tau_3}{a_1a_2} + d_{31}E_2$$

$$\frac{\partial v_1}{\partial\gamma} + \frac{g_1}{a_1} = s_{66}^E\frac{\tau_{13}}{a_2}, \quad \frac{\partial v_2}{\partial\gamma} + \frac{g_2}{a_2} = s_{44}^E\frac{\tau_{23}}{a_1} + d_{15}E_3$$

$$D_1 = \varepsilon_{11}^T E_1 + d_{15}\frac{\tau_{12}}{a_2}$$

$$D_2 = \varepsilon_{33}^T E_2 + d_{31}\left(\frac{\tau_1}{a_2} + \frac{\tau_3}{a_1a_2}\right) + d_{33}\frac{\tau_2}{a_1}$$

$$D_3 = \varepsilon_{11}^T E_3 + d_{15}\frac{\tau_{23}}{a_1}, \quad k_i = \frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}, \quad a_i = 1 + \frac{\gamma}{R_i}$$

The electrostatic equations

$$\frac{\partial D_3}{\partial\gamma} + \frac{1}{A_1A_2}\frac{\partial(A_2D_1)}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial(A_1D_2)}{\partial\alpha_2} = 0 \tag{1.2}$$

$$E_3 = -\frac{\partial\psi}{\partial\gamma}, \quad E_i = -\frac{1}{A_i}\frac{\partial\psi}{\partial\alpha_i}$$

The strain-displacement formulas

$$e_i = \frac{1}{A_i}\frac{\partial v_i}{\partial\alpha_i} + k_iv_j + \frac{v_3}{R_i}, \quad m_i = \frac{1}{A_j}\frac{\partial v_i}{\partial\alpha_j} - k_jv_j \tag{1.3}$$

$$g_i = \frac{1}{A_i}\frac{\partial v_3}{\partial\alpha_i} - \frac{v_i}{R_i}$$

Here and henceforth, two different equations can be obtained from each with subscripts i and j, one by setting $i=1, j=2$, and the other by setting $i=2, j=1$.

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The nonsymmetric tensor associated with the symmetric tensor σ_{ij}, σ_{i3}, σ_{33} by the formulas

$$\tau_i = a_j\sigma_{ii}, \quad \tau_{ij} = a_i\sigma_{ij}, \quad \tau_{i3} = a_j\sigma_{i3}, \quad \tau_3 = a_1a_2\sigma_{33} \tag{1.4}$$

is introduced in (1.1).

To shorten the writing in the piezoeffect equations, the following notation is introduced for the constants characterizing the electrical and mechanical properties of the shell material:

$$n_{11} = \frac{s_{33}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2}, \quad n_{12} = n_{21} = -\frac{s_{13}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2} \tag{1.5}$$

$$n_{22} = \frac{s_{11}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2}, \quad p_1 = \frac{s_{12}^E s_{33}^E - (s_{13}^E)^2}{s_{11}^E s_{33}^E - (s_{13}^E)^2}, \quad p_2 = \frac{s_{11}^E s_{13}^E - s_{12}^E s_{13}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2}$$

$$c_1 = \frac{d_{31}s_{33}^E - d_{33}s_{13}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2}, \quad c_2 = \frac{d_{33}s_{11}^E - d_{31}s_{13}^E}{s_{11}^E s_{33}^E - (s_{13}^E)^2}$$

The notation in (1.1)—(1.5) is v_i, v_3 are displacements, A_i are coefficients of the first quadratic form of the middle surface, R_i are its principal radii of curvature, s_{11}^E, s_{12}^E, s_{13}^E, s_{33}^E, s_{44}^E are the elastic compliances for a zero electrical field, d_{31}, d_{15}, d_{33} are piezoelectrical constants, ε_{11}^T, ε_{33}^T are the dielectric permittivities for zero voltages, $\mathbf{E}$ is the electrical field intensity vector, and $\mathbf{D}$ is the electrical induction vector.

The equilibrium equations have the same form as in the theory of nonelectrical shells, hence, we construct just the electroelasticity relationships.

2. Let us examine a shell not covered with electrodes, on whose facial surfaces the following surface load is given

$$\left.\frac{\tau_3}{a_1a_2}\right|_{\gamma=\pm h} = \pm q_3^{\pm}, \left.\frac{\tau_{i3}}{a_j}\right|_{\gamma=\pm h} = \pm q_i^{\pm} \tag{2.1}$$

In the no-electrode case, the electrical conditions have the following form:

$$D_3 |_{\gamma=\pm h} = 0 \tag{2.2}$$

To obtain conditions on the external shell surface, just the plus sign of the double signs in (2.1) and (2.2) must be taken, since conditions on the inner surface are obtained if the minus is taken.

We take the following asymptotic representation for the desired quantities of the electroelastic state under consideration:

$$\begin{gathered} v_i = \eta^s v_{i*}, \ v_3 = \eta^c v_{3*}, \ \tau_i = \tau_{i*}, \ \tau_{ij} = \tau_{ij*} \\ \tau_3 = \eta^{1-c}\tau_{3*}, \ \tau_{i3} = \eta^{1-s}\tau_{i3*}, \ E_3 = \eta^{1-s}E_{3*} \\ E_i = E_{i*}, \ D_3 = \eta^{2-3s+c}D_{3*}, \ D_i = D_{i*}, \ \psi = \eta^{2-s}\psi_* \\ c = 0, \quad 0 \leqslant s < 1/2; \quad c = -1 + 2s, \quad 1/2 \leqslant s < 1 \end{gathered} \tag{2.3}$$

Here η is the relative semi-thickness of the shell, and s, according to the terminology used in /1/, is the index of variability of the electroelastic state.

By means of (2.3) the desired quantities are replaced by quantities with asterisks which have the identical asymptotic order as $\eta \to 0$. The asymptotic representation (2.3) taken results in a non-contradictory theory in a first approximation. Moreover, (2.3) are confirmed in simple solutions /3/.

Let us execute the scale expansion along coordinate lines usual for asymptotic methods in the piezoelasticity equations (R is the shell characteristic dimension):

$$\alpha_i = \eta^s R\xi_i, \ \gamma = \eta^1 R\zeta \tag{2.4}$$

The coordinates ξ_i and ζ are introduced in such a manner that differentiation with respect to them would not result in a substantial increase in the functions desired. Formulas (2.4) mean that the desired quantities will increase η^{-s}-fold under differentiation with respect to the coordinates α_i and η^{-1}-fold under differentiation with respect to γ.

We substitute (2.3) and (2.4) into (1.1)—(1.4). Consequently, we obtain the following equations

$$\tau_{i*} = n_{ii}\frac{a_j}{a_i}\frac{1}{R}e_{i*} + n_{ij}\frac{1}{R}e_{j*} - \eta^{1-c}p_i\frac{\tau_{3*}}{a_i} - c_ia_jE_{3*} \tag{2.5}$$

$$
\begin{aligned}
&\tau_{ij*} = \frac{1}{s_{44}^E}\frac{1}{R}\left(\frac{a_i}{a_j} m_{i*} + m_{j*}\right) - \frac{d_{15}}{s_{44}^E} a_i E_{1*} \\
&\frac{1}{R}\frac{\partial v_{3*}}{\partial \zeta} = \eta^{1-c}\left(s_{12}^E \frac{\tau_{1*}}{a_2} + s_{13}^E \frac{\tau_{2*}}{a_1}\right) + \eta^{2-2c} s_{11}^E \frac{\tau_{3*}}{a_1 a_2} + \eta^{1-c} d_{31} E_{2*} \\
&\frac{\partial v_{1*}}{\partial \zeta} + \eta^{1-2s+c}\frac{g_{1*}}{a_1} = \eta^{2-2s} s_{66}^E R \frac{\tau_{13*}}{a_2} \\
&\frac{\partial v_{2*}}{\partial \zeta} + \eta^{1-2s+c}\frac{g_{2*}}{a_2} = \eta^{2-2s} s_{44}^E R \frac{\tau_{23*}}{a_1} + \eta^{2-2s} R d_{15} E_{3*} \\
&D_{1*} = \varepsilon_{11}^T E_{1*} + d_{15}\frac{\tau_{12*}}{a_1} \\
&\eta^{1-2s+c} D_{3*} = \varepsilon_{11}^T E_{3*} + d_{15}\frac{\tau_{23*}}{a_1} \\
&D_{2*} = \varepsilon_{33}^T E_{2*} + d_{31}\left(\frac{\tau_{1*}}{a_2} + \eta^{1-c}\frac{\tau_{3*}}{a_1 a_2}\right) + d_{33}\frac{\tau_{2*}}{a_1} \\
&\eta^{1-2s+c}\frac{\partial D_{3*}}{\partial \xi} + \frac{1}{A_1 A_2}\frac{\partial (A_2 D_{1*})}{\partial \xi_1} + \frac{1}{A_1 A_2}\frac{\partial (A_1 D_{2*})}{\partial \xi_2} = 0 \\
&E_{3*} = -\frac{1}{R}\frac{\partial \psi_*}{\partial \zeta}, \quad E_{i*} = -\frac{1}{R}\frac{1}{A_i}\frac{\partial \psi_*}{\partial \xi_i} \\
&e_{i*} = \frac{1}{A_i}\frac{\partial v_{i*}}{\partial \xi_i} + \eta^s R k_i v_{j*} + \eta^c \frac{R}{R_i} v_{3*} \\
&m_{i*} = \frac{1}{A_j}\frac{\partial v_{i*}}{\partial \xi_j} - \eta^s R k_j v_{j*}, \quad g_{i*} = \frac{1}{A_i}\frac{\partial v_{3*}}{\partial \xi_i} - \eta^{2s-c}\frac{R}{R_i} v_{i*}
\end{aligned}
$$

The order of each term of the equation with respect to the principal terms in the same equation is given, in the formulas written down, by the factor η to a non-negative power, ahead of it.

We construct the two-dimensional equations of state to the accuracy of quantities of order ε where

$$\varepsilon = O(\eta^{2-2s}) \tag{2.6}$$

Integrating (2.5) successively with respect to ζ to the accuracy of quantities of order (2.6), we obtain an expansion of the following form:

$$P_* = \sum_{i=0}^{n} \zeta^i a_i P_{,i} \tag{2.7}$$

where P_* should be understood to be any of the desired quantities $v_{i*}, \ldots, \psi_*$ and n, a_i take the following values for each of these quantities

$$
\begin{aligned}
&a_0 = 1;\ n = 1,\ a_1 = n^{1-2s+c} \text{ for } v_{i*}, e_{i*}, m_{i*}, \tau_{i*}, \tau_{ij*}, D_{i*} \\
&n = 1,\ a_1 = \eta^{1-c} \text{ for } v_{3*}, g_{i*},\ n = 0 \text{ for } E_{i*} \\
&n = 2,\ a_1 = 1,\ a_2 = \eta^{1-2s+c} \text{ for } v_{i3*}, E_{3*} \\
&n = 3,\ a_1 = 1,\ a_2 = \eta^{1-2s+c},\ a_3 = \eta^{2-4s+2c} \text{ for } \tau_{3*} \\
&n = 3,\ a_1 = a_3 = \eta^{1-c},\ a_2 = 1 \text{ for } D_{3*} \\
&n = 3,\ a_1 = a_2 = \eta^{2-2s},\ a_3 = \eta^{3-4s+c} \text{ for } \psi
\end{aligned}
\tag{2.8}
$$

Let us substitute the expansions (2.7) and (2.8) into (2.5). After equating coefficients of identical powers of ζ we obtain the following equations

$$
\begin{aligned}
&\tau_{i,0} = \frac{1}{R}(n_{ii} e_{i,0} + n_{ij} e_{j,0}) - \eta^{1-c} p_i \tau_{3,0} - c_i E_{2,0} \\
&\tau_{ij,0} = \frac{1}{R}\frac{1}{s_{44}^E}(m_{i,0} + m_{j,0}) - \frac{d_{15}}{s_{44}^E} E_{1,0} \\
&\tau_{i,1} = \frac{1}{R}(n_{ii} e_{i,1} + n_{ij} e_{j,1}) + \eta^{2s-c} n_{ii}\left(\frac{1}{R_j} - \frac{1}{R_i}\right) e_{i,0} - \eta^{2s-2c} p_i \tau_{3,1} - \eta^{2s-c} c_i \frac{R}{R_j} E_{2,0} \\
&\tau_{ij,1} = \frac{1}{R}\frac{1}{s_{44}^E}(m_{i,1} + m_{j,1}) + \eta^{2-s-c}\frac{1}{s_{44}^E}\left(\frac{1}{R_i} - \frac{1}{R_j}\right) m_{i,0} - \eta^{2s-c}\frac{d_{15}}{s_{44}^E}\frac{R}{R_i} E_{1,0} \\
&v_{3,1} = R(s_{12}{}^E \tau_{1,0} + s_{13}{}^E \tau_{2,0}) + d_{31} R E_{2,0}, \quad v_{i,1} = -g_{i,0} \\
&D_{1,0} = \varepsilon_{11}^T F_{1,0} + d_{15}\tau_{12,0}, \quad D_{1,1} = d_{15}\left(\tau_{12,1} - \eta^{2s-c}\frac{R}{R_1}\tau_{12,0}\right) \\
&D_{2,0} = \varepsilon_{33}{}^T E_{2,0} + d_{31}\tau_{1,0} + d_{33}\tau_{2,0} + \eta^{1-c} d_{31}\tau_{3,0} \\
&D_{2,1} = d_{31}\tau_{1,1} + d_{33}\tau_{2,1} - \eta^{2s-c}\left(d_{31}\frac{R}{R_2}\tau_{1,0} + d_{33}\frac{R}{R_1}\tau_{2,0}\right) + \eta^{2s-2c} d_{31}\tau_{3,1}
\end{aligned}
\tag{2.9}
$$

$$\eta^{1-2s+c} D_{3,0} = \varepsilon_{11}^T E_{3,0} + d_{15}\tau_{23,0}$$
$$\varepsilon_{11}^T E_{3,1} + d_{15}\tau_{23,1} - \eta^1 d_{15}\frac{R}{R_1}\tau_{23,0} = 0$$
$$D_{3,2} = \varepsilon_{11}^T E_{3,2} + d_{15}\tau_{23,2} - \eta^{2s-c} d_{15}\frac{R}{R_1}\tau_{23,1}$$

$$\frac{\partial (A_2 D_{1,0})}{\partial \xi_1} + \frac{\partial (A_1 D_{2,0})}{\partial \xi_2} = 0$$
$$D_{3,2} = -\frac{1}{2A_1A_2}\left[\frac{\partial (A_2 D_{1,1})}{\partial \xi_1} + \frac{\partial (A_1 D_{2,1})}{\partial \xi_2}\right]$$
$$E_{i,0} = -\frac{1}{A_i}\frac{\partial \psi_{,0}}{\partial \xi_i}, \quad E_{3,k} = -\frac{k+1}{R}\psi_{,k+1} \quad (k = 0, 1, 2)$$
$$\tau_{i3,0} + \eta^{1-2s+c}\tau_{i3,2} = \frac{\eta^{-1+s}}{2}\left[q_i^+ - q_i^- + \eta^1\frac{R}{R_j}(q_i^+ + q_i^-)\right]$$
$$\tau_{i3,1} = \frac{\eta^{-1+s}}{2}\left[q_i^+ + q_i^- + \eta^1\frac{R}{R_j}(q_i^+ - q_i^-)\right]$$
$$\tau_{3,0} + \eta^{1-2s+c}\tau_{3,2} = \frac{\eta^{-1+c}}{2}\left[q_3^+ - q_3^- + \eta^1\left(\frac{R}{R_1} + \frac{R}{R_2}\right)(q_3^+ + q_3^-)\right]$$
$$\tau_{3,1} + \eta^{2-4s+2c}\tau_{3,3} = \frac{\eta^{-1+c}}{2}\left[q_3^+ + q_3^- + \eta^1\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(q_3^+ - q_3^-)\right]$$

The quantities $e_{i,0}$, $g_{i,0}$, $m_{i,0}$, $m_{i,1}$ are found according to (2.5) in which the asterisks should be replaced by zero or one, respectively, while the following formulas hold for $e_{i,1}$, $g_{i,1}$:

$$e_{i,1} = \frac{1}{A_i}\frac{\partial v_{i,1}}{\partial \xi_i} + \eta^s R k_i v_{j,1} + \eta^{2s-c} R\frac{v_{3,1}}{R_i}$$
$$g_{i,1} = \frac{1}{A_i}\frac{\partial v_{3,1}}{\partial \xi_i} - \eta^c R\frac{v_{i,1}}{R_i}$$

Let us turn to quantities used in the theory of shells in the equations obtained. Using (2.7) and (2.3), we find that the displacements u_i, w of the shell middle surface are related to the three-dimensional displacements as follows

$$u_i = v_i|_{\zeta=0} = \eta^s v_{i,0}, \quad w = -v_3|_{\zeta=0} = -\eta^c v_{3,0} \tag{2.10}$$

The forces and moments are expressed in terms of the stresses by using (2.7), (2.8) and (2.3):

$$T_i = \int_{-h}^{+h}\tau_i\, d\gamma = 2h\tau_{i,0}, \quad S_{ij} = \int_{-h}^{+h}\tau_{ij}\, d\gamma = 2h\tau_{ij,0} \tag{2.11}$$
$$G_i = -\int_{-h}^{+h}\tau_i\gamma\, d\gamma = -\eta^{1-2s+c}\frac{2h^3}{3}\tau_{i,1}$$
$$H_{ij} = \int_{-h}^{+h}\tau_{ij}\gamma\, d\gamma = \eta^{1-2s+c}\frac{2h^3}{3}\tau_{ij,1}$$
$$N_i = -\int_{-h}^{+h}\tau_{i3}\, d\gamma = -\eta^{1-s}2h\left(\tau_{i3,0} + \frac{\eta^{1-2s+c}}{3}\tau_{i3,2}\right)$$

Using (2.10), (2.3), (2.9), we express the quantities $v_{i,1}$, $v_{3,1}$, $e_{i,0}, \ldots, e_{i,1}$ in terms of the angles of rotation and the strain components of the middle surface

$$e_{i,0} = R\varepsilon_i, \quad m_{i,0} = R\omega_j, \; g_{i,0} = \eta^{-1-c}\gamma_i \tag{2.12}$$
$$v_{i,1} = -\eta^{s-c}R\gamma_i, \; v_{3,1} = R\,(s_{12}^E n_{11} + s_{13}^E n_{21})\varepsilon_1 + R\,(s_{12}^E n_{12} + s_{13}^E n_{22})\varepsilon_2 + RE_2^{(0)}(d_{31} - s_{12}^E c_1 - s_{13}^E c_2)$$
$$m_{i,1} = \eta^{-c+2s}R^2\left(\tau - \frac{\omega_i}{R_j}\right)$$
$$e_{i,1} = \eta^{-c+2s}R^2\varkappa_i + \eta^{-c+2s}\frac{R^2}{R_i}\left[(s_{12}^E n_{11} + s_{13}^E n_{21})\varepsilon_1 + (s_{12}^E n_{12} + s_{13}^E n_{22})\varepsilon_2 + E_2^{(0)}(d_{31} - s_{12}^E c_1 - s_{13}^E c_2)\right]$$

The angles of rotation ω_j, γ_i and the middle-surface strain components ε_i, ω, $\varkappa_i$, τ are expressed in terms of the middle-surface displacements by means of formulas from /1/.

As result of transforming (2.9) and taking account of (2.11) and (2.12), we obtain formulas in the shell theory terminology, which we separate into two groups. Among the first group are the following

$$
\begin{aligned}
&T_i = 2h\,(n_{ii}\varepsilon_i + n_{ij}\varepsilon_j) - 2hc_iE_2^{(0)} - \{hp_i\,(q_3^+ - q_3^-)\} \qquad (2.13)\\
&S_{ij} = \frac{2h}{s_{44}^E}(\omega - d_{15}E_1^{(0)})\\
&G_i = -\frac{2h^3}{3}(n_{ii}\varkappa_i + n_{ij}\varkappa_j) + \Big\{-\frac{2h^3}{3}\Big(\frac{n_{ii}}{R_i} + \frac{n_{ij}}{R_j}\Big)[(s_{12}^E n_{11} +\\
&\quad s_{13}^E n_{21})\,\varepsilon_1 + (s_{12}^E n_{12} + s_{13}^E n_{22})\,\varepsilon_2 + (d_{31} - s_{12}^E c_1 - s_{13}^E c_2)\,E_2^{(0)}] -\\
&\quad \frac{2h^3}{3} n_{ii}\Big(\frac{1}{R_j} - \frac{1}{R_i}\Big)\varepsilon_i + \frac{2h^3}{3} c_i \frac{E_2^{(0)}}{R_j} + \frac{h^2}{3} p_i\,(q_3^+ + q_3^-)\Big\}\\
&H_{ij} = \frac{2h^3}{3s_{44}^E}\tau - \Big\{\frac{h^3}{3s_{44}^E}\frac{1}{R_i}(\omega + 2d_{15}E_1^{(0)}\Big\}\\
&E_i^{(0)} = -\frac{1}{A_i}\frac{\partial\psi^{(0)}}{\partial\alpha_i}, \quad D_1^{(0)} = \varepsilon_{11}^T E_1^{(0)} + \frac{d_{15}}{2h} S_{12}\\
&D_2^{(0)} = \varepsilon_{33}^T E_2^{(0)} + \frac{d_{31}}{2h} T_1 + \frac{d_{33}}{2h} T_2 + \Big\{\frac{d_{31}}{2}(q_3^+ - q_3^-)\Big\}\\
&\frac{\partial\,(A_2D_1^{(0)})}{\partial\alpha_1} + \frac{\partial\,(A_1D_2^{(0)})}{\partial\alpha_2} = 0
\end{aligned}
$$

Here and henceforth, the quantities with numerical superscript in parentheses are the coefficients of the desired quantities in expansions in powers of γ. These expansions are obtained if ζ is replaced by γ in (2.7) and (2.8) by using (2.4) and taking (2.3) into account. They have the form

$$P = \sum_{i=0}^{n} \gamma^i P^{(i)} \qquad (2.14)$$

where the number n is determined by (2.8) for each of the desired quantities. Small terms of the order of η^1 in comparison to the principal terms are in braces everywhere.

In addition to (2.13), the equilibrium equations and strain-displacement formulas should be included in (2.13). We consequently obtain a closed system of tenth order differential equations in the unknown mechanical and electrical quantities. We refer all the remaining equations to the second group

$$
\begin{aligned}
&D_1^{(1)} = \frac{3d_{15}}{2h^3} H_{12} - \Big\{\frac{d_{15}}{2h}\frac{S_{12}}{R_1}\Big\} \qquad (2.15)\\
&D_2^{(1)} = -\frac{3}{2h^3}(d_{31}G_1 + d_{33}G_2) + \Big\{-\frac{1}{2h}\Big(d_{31}\frac{T_1}{R_2} + d_{33}\frac{T_2}{R_1}\Big) + \frac{d_{31}}{2h}(q_3^+ + q_3^-)\Big\}\\
&D_3^{(0)} = -h^2 D_3^{(2)} = \frac{h^2}{2A_1A_2}\Big[\frac{\partial\,(A_2D_1^{(1)})}{\partial\alpha_1} + \frac{\partial\,(A_1D_2^{(1)})}{\partial\alpha_2}\Big]\\
&E_3^{(0)} = \frac{1}{\varepsilon_{11}{}^T} D_3^{(0)} + \frac{d_{15}}{4\varepsilon_{11}{}^T}\Big[\frac{3N_2}{h} + (q_2^+ - q_2^-) + \Big\{\frac{h}{R_1}(q_2^+ + q_2^-)\Big\}\Big]\\
&E_3^{(1)} = -\frac{d_{15}}{2h\varepsilon_{11}{}^T}(q_2^+ + q_2^-) - \Big\{\frac{3d_{15}}{4\varepsilon_{11}{}^T}\frac{1}{R_1}\Big(\frac{N_2}{h} + q_2^+ - q_2^-\Big)\Big\}\\
&E_3^{(2)} = \frac{1}{\varepsilon_{11}{}^T} D_3^{(2)} - \frac{3D_{15}}{4\varepsilon_{11}{}^T h^2}\Big(\frac{N_2}{h} + q_2^+ - q_2^-\Big) + \Big\{\frac{d_{15}}{\varepsilon_{11}{}^T}\frac{1}{R_1}\frac{q_2^+ + q_2^-}{2h}\Big\}\\
&\psi^{(k+1)} = -\frac{1}{k+1} E_3^{(k)} \quad (k = 0, 1, 2)
\end{aligned}
$$

According to (2.15), after the solution of the system of equations of the first group has been found, the electrical quantities not in the first group can be predetermined by using direct operations.

Let us note that the complete problem does not generally allow separation into mechanical and electrical problems. Some particular cases are the exception, for instance, for the axisymmetric problem /3/, all the electrical quantities in (2.15) can successfully be expressed in terms of forces, whereupon the complete problem separates into mechanical and electrical problems. In this case the equations of the mechanical problem differ from the equations of the theory of non-electrical shells only by the meaning of the coefficients in the equations of state adhead of the strain components.

3. Let us examine a piezoelectric shell with facial surfaces covered completely with electrodes on which the value V of the potential is given V (V is a function only of the time)

$$\psi\,|_{\gamma=\pm h} = \pm V \tag{3.1}$$

It is assumed that there is no mechanical surface load

$$\left.\frac{\tau_3}{a_1 a_2}\right|_{\gamma=\pm h} = 0, \quad \left.\frac{\tau_{i3}}{a_j}\right|_{\gamma=\pm h} = 0 \tag{3.2}$$

We take the following asymptotic representation for the desired quantities of the electroelastic state

$$\begin{aligned} &v_1 = \eta^{1-s+c} v_{1*}, \ v_2 = v_{2*}, \ O\,(v_2\,|_{\gamma=0}) = \eta^{1-s+c} \\ &v_3 = \eta^{1-2s+2c} v_{3*}, \ \tau_i = \tau_{i*}, \ \tau_{ij} = \tau_{ij*} \\ &\tau_{i3} = \eta^{1-s}\tau_{i3*}, \ \tau_3 = \eta^{1-c}\tau_{3*}, \ E_3 = \eta^{-1}E_{3*} \\ &\psi = \psi_*, \ D_i = D_{i*}, \ D_3 = \eta^{-1}D_{3*} \end{aligned} \tag{3.3}$$

where c, s, η have the same meaning as before. In the case under consideration, it follows from (1.2) and (3.1) that the tangential electrical field intensity vector components E_1 and E_2 are zero.

Let us make the change of variables (2.4) by substituting the asymptotic (3.3) into the initial equations, we then integrate with respect to ζ to the accuracy of (2.6), whereupon we obtain an expansion of the form (2.7), of which we present here only the formulas for the displacements and the principal stresses

$$\begin{aligned} &v_{1*} = v_{1,0} + \eta^{1-2s+c}\zeta v_{1,1}, \ v_{2*} = \eta^{1-s+c} v_{2,0} + \zeta U_{2,1} + \eta^1\zeta^2 v_{2,2} \\ &v_{3*} = \vartheta_{3,0} + \eta^{1-c}\zeta v_{3,1} + \lambda^{2s-2c}\zeta^2 v_{3,2} \\ &\tau_{i*} = \eta^{1-2s+c}\tau_{i,0} + \zeta\tau_{i,1} + \eta^{1-c}\zeta^2\tau_{i,2} \\ &\tau_{ij*} = \eta^{1-2s+c}\tau_{ij,0} + \zeta\tau_{ij,1} + \eta^1\zeta^2\tau_{ij,2} \end{aligned} \tag{3.4}$$

Comparing (3.4) with the analogous formulas of the theory of non-electrical shells /1/, we note that the expansions obtained for the stresses have almost the same form as the appropriate expansions in the case of a pure bending state. The formulas for the displacements v_{2*} and v_{3*} have no analogs in the theory of non-electrical shells. The greatest of the displacement components is $v_{2,1}$. This is explained by the fact that the domains oriented along the lines α_2 tend to occupy a position normal to the middle surface upon imposition of the electrical field, consequently, the quantities $\tau_{i,1}$, $\tau_{ij,1}$ by which the moments are determined, are maximal in the principal stress. Performing the computations in the same sequence as in Sect.2, we obtain the following electroelasticity relationships for the shell with electrodes:

$$\begin{aligned} &T_i = 2h\,(n_{ii}\varepsilon_i + n_{ij}\varepsilon_j) + \left\{\frac{h^3}{3}k_1 d_{15}E_3\left[\frac{n_{i1}}{R_2} + \frac{n_{i2}}{k_1}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{1}{R_2} + \right.\right. \\ &\left.\left.(s_{12}{}^E n_{11} + s_{13}{}^E n_{21})\left(\frac{n_{ii}}{R_i} + \frac{n_{ij}}{R_j}\right) + 2n_{ii}\left(\frac{1}{R_2} - \frac{1}{R_i}\right) + 2p_i\left(\frac{n_{11}}{R_1} + \frac{n_{21}}{R_2}\right)\right]\right\} \\ &G_i = -\frac{2h^3}{3}(n_{ii}\varkappa_i + n_{ij}\varkappa_j + n_{i1}k_1 d_{15}E_3) \\ &S_{21} = \frac{2h}{s_{44}{}^E}\omega + \left\{\frac{h^3 d_{15}E_3}{3s_{44}{}^E}\left(-\frac{k_2}{R_2} + \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\frac{1}{R_2}\right)\right\} \\ &S_{12} = S_{21} - \left\{\frac{2h^3}{3s_{44}{}^E}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)(d_{15}k_2E_3 - \tau)\right\} \\ &H_{12} = H_{21} = \frac{4h^3}{3s_{44}{}^E}\tau - \frac{3h^3k_2d_{15}}{3s_{44}{}^E}E_3, \quad E_3 = -\frac{V}{h} \end{aligned} \tag{3.5}$$

Together with the equilibrium equations and the strain-displacement formulas, the relationships (3.5) comprise a close system of eighth-order differential equations that do not contain unknown electrical quantities. Since the Kirchhoff–Love hypotheses are not satisfied for the theory constructed, the formulas to go from the forces and moment to the stresses and from the middle surface displacements to the three-dimensional displacements are qualitatively different from the corresponding formulas of the classical theory of non-electrical shells. The formulas (2.14) hold for the expansions of the desired quantities in the variable γ where the number n is given in the following manner:

$$n = 1 \text{ for } v_1;\ n = 0 \text{ for } E_3,\ D_3$$
$$n = 2 \text{ for } v_2,\ v_3,\ \tau_i,\ \tau_{ij},\ D_i;\ n = 3 \text{ for } \tau_{i3},\ \tau_3$$

After the mechanic problem has been solved in theory of shell terminology, it is possible to go over to the three-dimensional desired quantities by using the following formulas:

$$v_i^{(0)} = u_i,\ v_3^{(0)} = -w,\ v_1^{(1)} = -\gamma_1$$
$$v_2^{(1)} = d_{15}E_3 - \gamma_2, \qquad \left\{v_2^{(2)} = \frac{d_{15}E_3}{2R_2}\right\}$$
$$\{v_3^{(1)} = (s_{12}{}^E n_{11} + s_{13}{}^E n_{21})\,\varepsilon_1 + (s_{12}{}^E n_{12} + s_{13}{}^E n_{22})\,\varepsilon_2\}$$
$$v_3^{(2)} = \frac{1}{2} k_1 d_{15} (s_{12}{}^E n_{11} + s_{13}{}^E n_{21}) E_3$$
$$\left\{\tau_1^{(2)} = \left[n_{11}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{n_{11}}{2R_2} + \frac{n_{12}}{2k_1}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{1}{R_2} + \frac{1}{2}\left(\frac{n_{11}}{R_1} + \frac{n_{21}}{R_2}\right)(s_{12}{}^E n_{11} + s_{13}{}^E n_{21} - p_1)\right] k_1 d_{15} E_3\right\}$$
$$\left\{\tau_2^{(2)} = \frac{1}{2}\left[\frac{n_{22}}{k_1}\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{1}{R_2} + (s_{12}{}^E n_{11} + s_{13}{}^E n_{21})\left(\frac{n_{22}}{R_2} + \frac{n_{21}}{R_1}\right) + \frac{n_{21}}{R_2} - p_2\left(\frac{n_{11}}{R_1} + \frac{n_{21}}{R_2}\right)\right] k_1 d_{15} E_3\right\}$$
$$\tau_i^{(0)} = \frac{T_i}{2h} - \left\{\frac{h^2}{3}\tau_i^{(2)}\right\},\quad \tau_i^{(1)} = -\frac{3}{2h^2} G_i$$
$$\left\{\tau_{12}^{(2)} = \frac{d_{15}E_3}{2s_{44}{}^E}\left[\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\frac{1}{R_2} + \frac{k_2}{R_2} - \frac{2k_2}{R_1}\right] - \frac{1}{s_{44}{}^E}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\tau\right\}$$
$$\left\{\tau_{21}^{(2)} = \frac{d_{15}E_3}{2s_{44}{}^E}\left\{\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\frac{1}{R_2} - \frac{k_1}{R_2}\right)\right\}$$
$$\tau_{ij}^{(0)} = \frac{S_{ij}}{2h} - \left\{\frac{h^2}{3}\tau_{ij}^{(2)}\right\},\quad \tau_{ij}^{(1)} = \frac{3}{2h^3} H_i$$
$$\tau_{i3}^{(0)} = -h^2\tau_{i3}^{(2)} = -\frac{3}{4h} N_i$$
$$\left\{\tau_{i3}^{(1)} = -h^2\tau_{i3}^{(3)} = -\left[\frac{1}{A_i}\frac{\partial\tau_i^{(0)}}{\partial\alpha_i} + \frac{1}{A_j}\frac{\partial\tau_{ij}^{(0)}}{\partial\alpha_j} + k_j(\tau_i^{(0)} - \tau_j^{(0)}) + k_i(\tau_{ij}^{(0)} + \tau_{ji}^{(0)})\right]\right\}$$
$$\tau_3^{(0)} = -h^2\tau_3^{(2)} = -\frac{h^2}{2}\left(\frac{n_{11}}{R_1} + \frac{n_{21}}{R_2}\right) k_1 d_{15} E_3$$
$$\tau_3^{(1)} = -h^2\tau_3^{(3)} = \frac{\tau_1^{(0)}}{R_1} + \frac{\tau_2^{(0)}}{R_2} - \frac{1}{A_1}\frac{\partial\tau_{13}^{(0)}}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial\tau_{23}^{(0)}}{\partial\alpha_2} - k_2\tau_{13}^{(0)} - k_1\tau_{23}^{(0)}$$

Then we calculate the electrical quantities by means of the stresses found:

$$D_3^{(0)} = \varepsilon_{11}{}^T E_3,\ D_1^{(0)} = d_{15}\tau_{12}^{(0)},\ D_1^{(1)} = d_{15}\tau_{12}^{(1)}$$
$$\left\{D_1^{(2)} = d_{15}\left(\tau_{12}^{(2)} - \frac{1}{R_2}\tau_{12}^{(1)}\right)\right\}$$
$$D_2^{(0)} = d_{31}\tau_1^{(0)} + d_{33}\tau_2^{(0)} + \{d_{31}\tau_3^{(0)}\}$$
$$D_2^{(1)} = d_{31}\tau_1^{(1)} + d_{33}\tau_2^{(1)}$$
$$\left\{D_2^{(2)} = d_{31}\tau_1^{(2)} + d_{33}\tau_2^{(2)} - \left(d_{31}\frac{\tau_1^{(1)}}{R_2} + d_{33}\frac{\tau_2^{(1)}}{R_1}\right) + d_{31}\tau_3^{(2)}\right\}$$

As before, those terms and equations that must be discarded are in the braces if an error of the order of η^1 is allowed in the theory.

It is seen from the formulas obtained that the complete problem separates into mechanical and electrical problems in the case under consideration, where the electrical quantities are calculated after the mechanical problem has been solved by using algebraic operations.

4. Electroelasticity relationships have been obtained above to the accuracy of quantities of the order of η^{2-2s}. It can be shown that this accuracy is optimal, as in the theory of non-electrical shells /1/: the theory is complicated qualitatively for an attempt at its improvement, the order of the system of differential equation rises, the need occurs for the introduction of an elasticity relationship connecting the transverse force and the transverse shear, etc.

The terms and equations in the braces permit raising the accuracy of the computation to quantities $O(\eta^{2-2s})$ for an electroelastic state with low variability $(s < 1/2)$, which is important in solving practical problems since medium-thickness piezoelectrical shells are used, as a rule, in engineering. For high indices of variability $(s \geqslant 1/2)$ the small terms in the braces are outside the accuracy taken, hence, they must be discarded.

If terms $O(\eta^1)$ are neglected in the electroelasticity relationships constructed, and simplifications are introduced that hold for the axisymmetric problem for meridian polarization, then we obtain the electroelasticity relations that agree with those deduced earlier in /3/.

Let us note that in this particular case, the simplified transfer formulas for shells covered completely by electrodes do not permit determination of the quantity $v_3^{(2)}$ (for $s = 0, v_{3,2}$ is of the same order as $v_{3,0}$). Moreover, for small s it is impossible to determine the stresses in terms of forces since the principal stresses along the normal coordinate vary according to a square law, and the formulas taken in shell theory provide only a linear law of stress variation over the thickness.

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